New Integer Coded Hexagonal QAM Schemes and Their Performance in AWGN Channel

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Abstract. Six coded modulation schemes based on integer codes for HQAM constellations with 16, 32, 64, 128, and 256 points are proposed. Their performance in the case of communication through AWGN channel have been studied. The presented results include comparisons in graphical form between probabilities for error per signal point (SER) in coded and uncoded case as well as probabilities for error per bit (BER) in the same cases.

1 Introduction

The notion coded modulation refers to any scheme that integrates modulation and error correcting techniques in unified process. Square quadrature amplitude modulation (SQAM) is maybe the most implemented in communication devices modulation scheme. Signal points of SQAM constellation form square lattice and the detection regions are also squares that simplifies detection procedure. However different types of constellations have been recently subject of research interest. One such constellation with 64 signal points is depicted in Fig. 1. It belongs to the family of so called *hexagonal quadrature amplitude modulation* (HQAM). The use of this name is motivated by the form of the most of detection regions that are hexagons. Signal points of the constellation form a lattice of equilateral triangles. This is the reason some authors name it Triangular QAM (TQAM). We shall use the name HQAM.

Various other constellations for detection and error correction, together with the methods developed for them, have been proposed and studied (see e.g. [1–6, 19]). Many of them are based on facts from algebraic number theory but their description is out of the goal of this paper. We are interested only in HQAM constellations.

As it is shown in [16] HQAM is more power efficient than SQAM preserving in general the low detection complexity of SQAM. In [17] a formula for the average energy per symbol of the HQAM is derived and approximate values of symbol error rate (SER) and bit error rate (BER) are calculated in the case of uncoded HQAM used in channel with additive white Gaussian noise (AWGN) .

The classical approach to coded modulation is the use of convolution codes for correcting erroneously detected signal points most often together with SQAM. An alternative approach is the use of integer codes. They enable lower complexity of decoding and can be designed to correct errors of types that are typical (most common) for the chosen channel and constellation. Coded modulation schemes based on SQAM and integer codes are studied in [11–13],

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In this paper we propose several new coded modulation schemes based on HQAM and integer codes. In [10] the reader can find description of a theoretical approach to computation of the performance of considered coded modulation schemes. Herein the performance results are obtained by computer simulation of the used proposed schemes for communication in AWGN channel. Hence the paper can be considered as a continuation of [10].

2 Preliminaries

2.1 HQAM Constellations

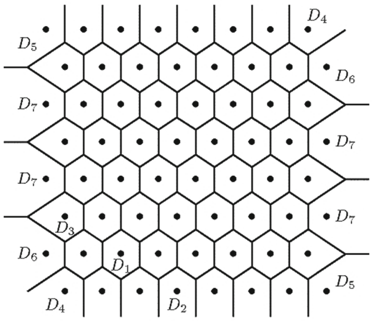


Fig. 1. 64-HTQAM constellation - uncoded case [10]

In this paper we consider HQAM constellations with $M = 16, 32, 64, 128$, and 256 signal points which form a lattice of congruent equilateral triangles with side equals $2d$. Signal points of such a constellation are placed in rows parallel to the horizontal axis and the constellation is symmetric with respect to the origin. The distance between horizontal rows is $h = d\sqrt{3}$ (which is the altitude of the equilateral triangles). It is shown in [17] that the power gain of considered M-ary HQAM over M-ary SQAM in decibels tends to 0.5799 dB when M tends to infinity. For example, the gain is 0.458, 0.5505, 0.5726 dB for $M = 16,\ 64,\ 256$, respectively. For a given M several constellations whose point form triangular lattice can be found in literature. They differ in the number of their rows and shapes but a little in their power efficiency. However this little improvement usually leads to more complex detection procedure due to increasing the number of different detection regions. The dominated form of detection regions is hexagon with side equals $a = 2d/\sqrt{3}$ (the regions of contour points are more complex). The 64-HQAM constellation with the different type of detection regions in uncoded case is presented in Fig. 1.

When M is even power of 2 constellations with $\sqrt{M}$ rows are maybe preferred since they have square shape and detection is similar to one of SQAM. For odd powers of 2 square shape constellation with removed points in corners are preferable.

2.2 Integer Codes

Let $\mathbb{Z}_A$ be the ring of non-negative integers modulo A, where A be a positive integer.

Definition 1. *An* ***integer code*** *of length n with parity-check matrix* $\mathbf{H} \in \mathbb{Z}_A^{m\times n}$ *is a subset of* $\mathbb{Z}_A^n$*, defined by*

$$\mathbf{C}(\mathbf{H}) = \{\mathbf{c} \in \mathbb{Z}_A^n \,|\, \mathbf{c}\mathbf{H}^T = \mathbf{0} \pmod{A}\}$$

Remark. When A is not prime $\mathbb{Z}_A$ is not a field. It is a ring with zero divisors and $\mathbb{Z}_A^n$ is a module over $\mathbb{Z}_A$. Integer codes are submodules, not linear subspaces, but the difference is not essential for our consideration and we take it into account when it is necessary.

Maybe the first application of integer codes was their use for correcting single insertion/deletion per codeword proposed by Varshamov and Tenengolz [18] in 1965. In 2004 [12] it was demonstrated that integer codes are very suitable for implementing in coded modulation schemes based on SQAM. Constructions of integer codes and discussions about their applications can be found also, in [7–9, 14, 15]

When a codeword $\mathbf{c}$ is sent through a communication channel (usually noisy) the received vector can be written in the form

$$\mathbf{r} = \mathbf{c} + \mathbf{e},$$

where $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbf{Z}_A^n$ is called error vector. An integer code C is referred to be ***t*-multiple** $(\pm d_1, \pm d_2, \ldots, \pm d_s)$***-error correctable*** if it enables the error vector $\mathbf{e}$ having up to t nonzero coordinates with values from the set $\{\pm d_1, \pm d_2, \ldots, \pm d_s\}$ to be recovered.

In this paper we use only codes with $m = 1$, i.e., codes with parity-check matrix

$$\mathbf{H} = (h_1, h_2, \ldots, h_n),\ h_i \in \mathbb{Z}_A.$$

Hence such a code C is a single $(\pm d_1, \pm d_2, \ldots, \pm d_s)$-error correctable if all syndromes $\{\pm d_i h_j \mid i = 1, \ldots,\ j = 1, \ldots, n\}$ are different which implies

$$2sn + 1 \leq A.$$

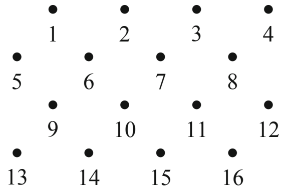
Codes that lie on this bound are called *perfect* in [11].

3 New Coded Modulation Schemes

When a signal point is sent through a channel not all signal points are equally probable to be erroneous detected as sent signal point by the receiver. The closest (up to 6 at distance $2d$ in HQAM) points are more probable. If signal points are labeled by elements of $\mathbb{Z}_A$ the aforesaid means that not all error vectors are equally probable. Some elements of $\mathbb{Z}_A$, for example $\{\pm d_1, \pm d_2, \ldots, \pm d_s\}$, appear more often as coordinates of error vector. This leads us to the following idea for constructing coded modulation scheme based on integer codes:

For a given constellation choose a labeling of signal points by elements of $\mathbb{Z}_A$ and a $(\pm d_1, \pm d_2, \ldots, \pm d_s)$-error correctable code in such a way that leads to smaller probability of error.

It is easy to check (by computer) that the proposed below codes have the described properties. But their finding is not so easy. They are found by combination of computer search and some theoretical arguments.



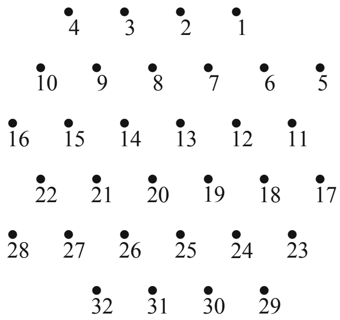


Fig. 2. 16-HQAM (left) and 32-HQAM (right) constellations

3.1 16-HQAM Scheme

Let us consider the integer code C over $\mathbb{Z}_{17}$ defined by a parity-check matrix $\mathbf{H} = (1,2)$. It easy to check that C is a single $(\pm1, \pm3, \pm4, \pm5)$-error correctable code, namely that the syndromes of all 16 vectors with one nonzero such coordinates are different. Hence C is a perfect code with length 2 and rate 1/2. We apply C to 16-HQAM with labeling as it is shown in Fig. 2. Note that there is no signal point for 0 but choosing only nonzero information symbols the parity check symbols are nonzero, too.

3.2 32-HQAM Scheme

Let us consider the 32-HQAM with the labeling given in Fig. 2. The constellation is obtained from 6×6 square shape 36-HQAM constellation by deleting its 4 corners. We propose two coded modulation schemes that are based on the labeling given in Fig. 2.

Scheme A Consider code over $\mathbb{Z}_{33}$ with parity-check matrix $\mathbf{H} = (2,1)$. It can correct any single error of type $(\pm1, \pm5, \pm6, \pm7)$. The encoding procedure is simple – any 5 bits are mapped into a nonzero element of $\mathbb{Z}_{33}$, i.e., into a signal point. Since $(2,33) = 1$ the check symbol is also nonzero. Hence the rate of the code is 1/2.

Scheme B In this coded modulation scheme we apply the integer code C over $\mathbb{Z}_{37}$ with parity-check matrix $\mathbf{H} = (4,3,2,1)$. It can correct any single error of type $(\pm1, \pm5, \pm6, \pm7)$ since the syndromes of all 32 vectors with one nonzero such coordinates are different. We can choose information symbols (first 3 coordinates) from 1 to 32 but the forth (parity check) symbol can take also the values 0, 33, 34, 35 and 36. However there are no signal points that correspond to these values. To solve the problems we reduce two times the number of used codewords to 2^{14} codewords whose coordinates take values only from 1 to 32. As a result the rate of C is slightly reduced from 4/5 to 14/20.

3.3 64-HQAM Scheme

Let us consider the square shape 64-HQAM constellation (see Fig. 1) with labeling similar to one of 16-HQAM, that is, starting from the leftmost point in the top row, labeling from left to right, till the rightmost point in the bottom row. We use the integer code C over $\mathbb{Z}_{65}$ defined by $\mathbf{H} = (12,6,5,4,3,2,1)$. The code can correct any single error with values $\{\pm 1, \pm 7, \pm 8, \pm 9\}$ that extends the detection region covering the nearest neighbor points. Since no signal point corresponds to 0 while all 65 elements of $\mathbb{Z}_{65}$ appear as value of the seventh (parity check) symbol we cannot use all 2^{36} codewords, that is, to map a binary block of length 36 to a codeword. Hence we have to reduce the number of used codewords to 2^{35} and during encoding to map block of length 35 to a codeword. Therefore the rate of proposed scheme is 35/42.

3.4 128-HQAM Scheme

The constellation (see Fig. 3) of the proposed coded modulation scheme is obtained from 12×12 HQAM constellation by deleting the leftmost and the rightmost points in the first four and the last four rows. Hence the four middle rows have 12, and the others rows 10 points. As usual the distance between points is $2d$, and between rows $h = d\sqrt{3}$. The average energy per point is $E_s = 72.5d^2$ that is quite better than $82d^2$ of the usually used constellation. The labeling is standard – from left to right and from the top to the bottom row.

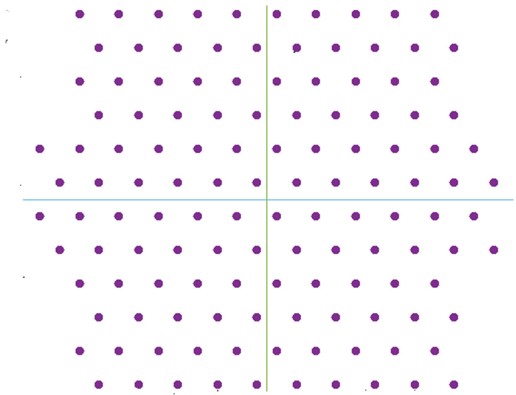


Fig. 3. 128-HQAM constellation

The code C applied to the considered scheme is over $\mathbb{Z}_{129}$ and with parity check matrix $\mathbf{H} = (31,5,3,2,1)$. The code C corrects any single error of type $(\pm 1, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13)$. The length of the code is 5 and the rate 27/35. (We have to lose one information bit in order to avoid appearing of 0 as a parity check symbol.)

3.5 256-HQAM Scheme

The proposed 256-HQAM coded modulation scheme is similar to ones for 16 and 64 signal points in sense that it has the same form and use the same principle of labeling. The proposed integer code C is over $\mathbb{Z}_{257}$, has block length 32, and a parity check matrix

$$\mathbf{H} = (1,\ldots,12,14,18,20,22,24,28,36,37,39,40,42,44,56,72,74,79,84,88,89,109).$$

The code corrects any single error of the type $(\pm1,\pm15,\pm16,\pm17)$. The rate of the code is 247/256 and it is perfect.

4 Performance of the Proposed Schemes

We consider an AWGN channel with power spectral density N_0. Hence if a signal point $\mathbf{s}$ is sent through the channel the received signal $\mathbf{r} = \mathbf{s} + \mathbf{n}$, where $\mathbf{n}$ is two dimensional zero-mean Gaussian random process with variance $\sigma^2 = N_0/2 = E_s/(2SNR)$, where E_s is the average energy of a constellation point and SNR is the signal to noise ratio per signal point (i.e. symbol). At the other end of the channel the detector estimates $\mathbf{r}$ and has to decide which signal point has been sent.

During computer simulations we stick to the following rule: We increase number of repetitions of experiments until at least the first nonzero cipher after decimal point becomes significant, that is, it does not change even several times increasing of number of repetitions.

The results of computer simulations present probabilities of error per bit (BER) and per signal point (SER) versus signal/noise ratio per bit E_b/N_0, where E_b is the average energy per bit.

4.1 16-HQAM Scheme

We simulate communication through AWGN channel using the coded modulation scheme described in Sect. 3.1. During our simulations we sent trough the channel 1 000 000 times any of 16 possible codewords (128 000 000 bits). As results we obtain the average probabilities of symbol error (SER) in uncoded and coded cases as well probability of bit error (BER) in coded case. For comparison we simulate communication using “Grey”correspondence between signal points and binary vectors which gives the best BER in uncoded case (see [10]). The results are given in Table 1. Figure 4 presents the probabilities of error per signal point.

Table 1. BER of 16-HQAM coded modulation scheme 3.1

E_b/N_0 dB	2	3	4	5	6	7	8	9	10	11
BER Grey	0.1099	0.0888	0.0684	0.0496	0.0333	0.0198	0.0107	0.0050	0.0018	0.0006
BER coded	0.0429	0.0284	0.0155	0.0120	0.0051	0.0017	0.0004	5.47e–05	4.69e–06	1.09e–06

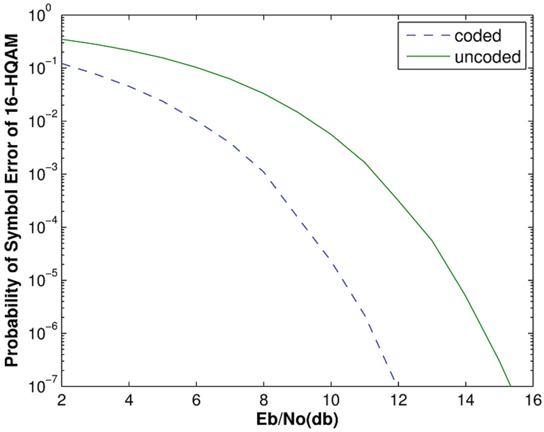


Fig. 4. SER of 16-HQAM scheme given in Sect. 3.1.

4.2 32-HQAM Scheme

Scheme A.The presented with Fig. 5 results are obtained by computer simulations of transmitting each of the 32 codewords tens billion times through AWGN channel for any value of SNR per bit from 3 to 18 dB.

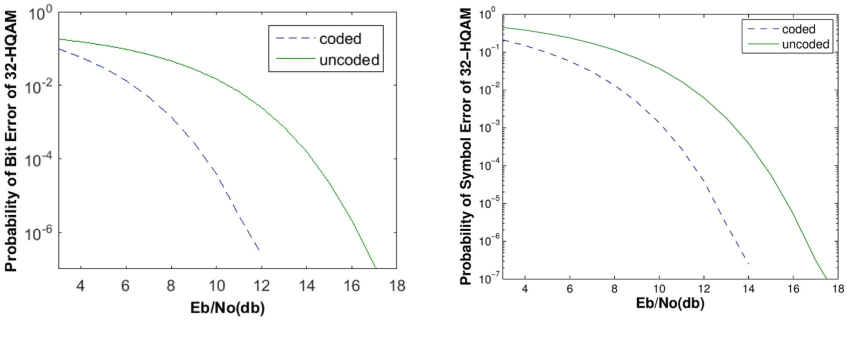


Fig. 5. BER and SER for 32-HQAM scheme over $\mathbb{Z}_{33}$

Scheme B The same simulations (for SNR per bit from 3 to 18) with coded modulation Scheme B give the performance presented in Fig. 6.

Scheme A demonstrates about 3 times better performance than Scheme B (it is difficult to be perceived in logarithmic scale) but this is the price payed for the 1.5 times higher rate of Scheme B.

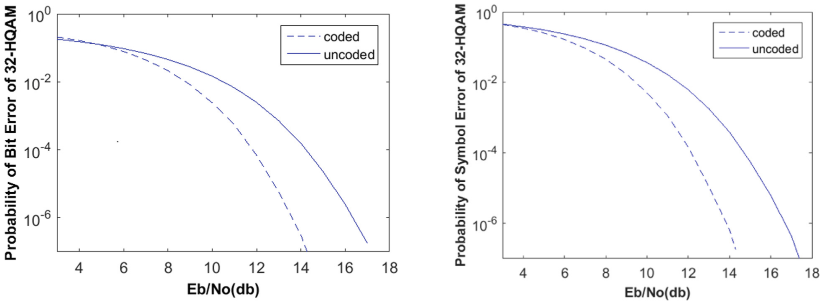


Fig. 6. BER and SER for 32-HQAM scheme over $\mathbb{Z}_{37}$

4.3 64-HQAM Scheme

For any value of SNR per bit from 9 to 20 dB we simulate a communication through AWGN channel with the chosen SNR based on the proposed in Sect. 3.3 coded modulation scheme. The presented results are obtained by sending 20 000 000 randomly generated codewords (840 000 000 bits). Of course we have made experiments with much longer sequences but the differences are less then 10^{-7}.

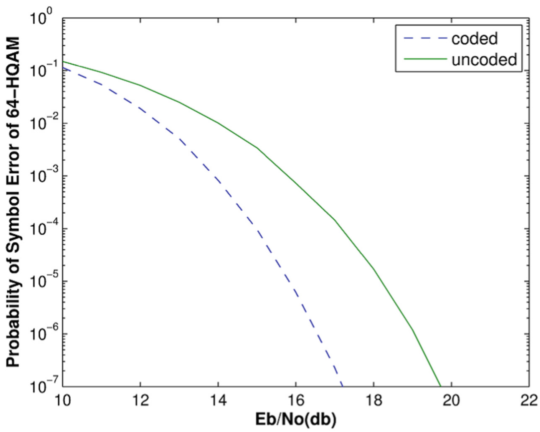


Fig. 7. SER of 64-HQAM scheme 3.3

For comparison we present the corresponding results for BER in the case of communication using Grey mapping.

4.4 128-HQAM Scheme

The results of the computer simulating communications based on the proposed in Sect. 3.4 coded modulation scheme are presented in Fig. 8.

Table 2. BER of 64-HQAM coded modulation scheme 3.3

E_b/N_0 dB	10	11	12	13	14	15	16	17	18	19
BER Grey	0.03124	0.01921	0.01106	0.00543	0.00222	0.00070	0.00017	3.6e-5	2.5e-6	2.8e-7
BER coded	0.04250	0.01906	0.00691	0.00181	0.00029	3.177e-5	8.098e-7	0	0	0

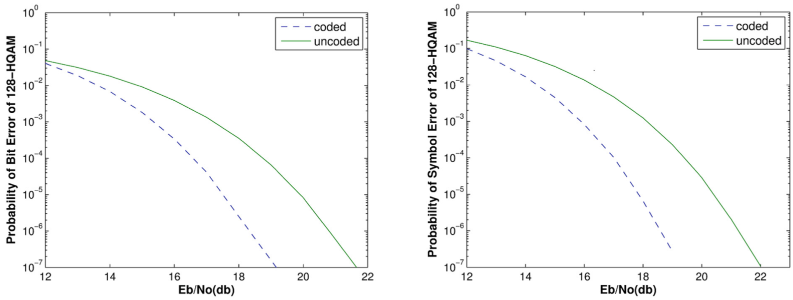


Fig. 8. BER and SER for 128-HQAM schemes over $\mathbb{Z}_{129}$

4.5 256-HQAM Scheme

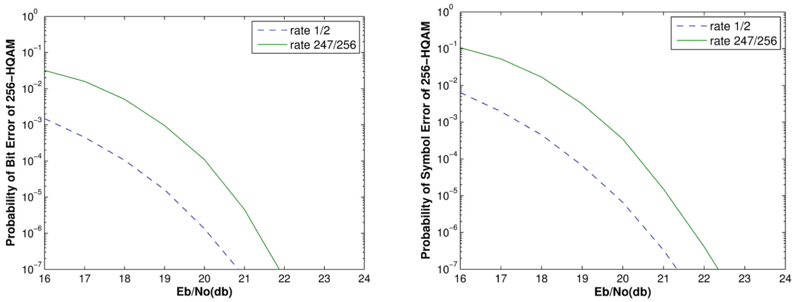


Fig. 9. BER and SER for two 256-HQAM schemes over $\mathbb{Z}_{257}$

In Fig. 9 we present the comparison between the performances of the code described in Sect. 3.5 and the code with parity check matrix $\mathbf{H} = (2, 1)$ studied in [10]. The difference is about 1 dB. It demonstrates the influence of the redundancy, thus the rate of the code, to the performance. The code given in Sect. 3.5 has worse performance for small values of SNR but the rate almost 1. But in the case of more pure (1 dB) channels the higher rate gives a big advantage.

5 Conclusion

In this paper we proposed six new coded modulation schemes that employ integer codes for error correction. Also the scheme with 128 points is based on a new more power efficient constellation. The proposed codes have standard and simple encoding and decoding.

Computer simulations of communication through AWGN channels using the proposed coded modulation schemes demonstrate their competitive performance. The presented results show improvement of about 3 dB in all of the considered cases. These performance results can also give a basic information to practitioners in their choice of coded modulation scheme.

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